Nonlinear theory of surface-wave–particle interactions in a cylindrical plasma

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(Received 15 December 1993)

This work is an application of the specular reflection hypothesis to the study of the nonlinear surface-wave–particle interactions in a cylindrical plasma. The model is based on nonlinear resolution of the Vlasov equation by the method of characteristics. The expression obtained for the rate of increase of kinetic energy per electron has permitted us to investigate the temporal behavior of nonlinear collisionless damping for different situations as a function of the critical parameters.

PACS number(s): 52.40.Db, 52.25.Dg

I. INTRODUCTION

In the last few years, great interest has been aroused in the study of surface-wave–particle propagation along a cylindrical plasma column. Early works by Trivelpiece and Gould [1,2] disclosed the propagation modes when a plasma is of finite transverse cross section, in the quasistatic approximation. In recent years most attention has been directed at the properties of the surface-wave sustained plasmas [3–6]. The principal properties and applications of this kind of plasma have been reviewed by Moisan and co-workers [7,8]. Prior to this paper [9] a theoretical model to analyze the linear surface-wave–particle interaction was developed. This model was based on the linear resolution of the Vlasov equation, with the specular reflection hypothesis. It is the purpose of this paper to provide a theoretical discussion of the nonlinear perturbation of the distribution function of a cylindrical plasma which is the support of a surface wave (Secs. III–IV). The knowledge of this distribution function permits us to calculate the nonlinear rate of increase of the kinetic energy per electron for collisionless damping (Sec. V). In Sec. II we briefly review the general theoretical considerations of surface-wave propagation along the cylindrical plasma. The paper concludes with a summary and conclusions (Sec. VI).

II. BASIC ASSUMPTIONS

Figure 1 shows the conventional scheme to obtain the propagating modes. The plasma tube is surrounded by a cylindrical metallic waveguide enclosing it completely. With regard to the wave, the plasma is a dielectric medium of permittivity ε_p,

$$\varepsilon_p = 1 - \left(\frac{\omega_p}{\omega}\right)^2, \tag{1}$$

where ω_p is the electron plasma angular frequency and ω is the wave angular frequency. The wave vector is directed parallel to the plasma column, whose axis coincides with the z direction.

The fields in the plasma ($0 < r < a$) are given by the expressions [10]

$$E_z = E_{0z}(r)\sin(\omega t - \beta z), \tag{2}$$
$$E_r = E_{0r}(r)\sin(\omega t - \beta z), \tag{3}$$
$$B_\phi = B_{0\phi}(r)\sin(\omega t - \beta z), \tag{4}$$

$$E_{0z} = EI_0(\Gamma r), \quad E_{0r} = \frac{\beta}{\Gamma^2}\frac{\partial E_{0z}}{\partial r}, \quad B_{0\phi} = \frac{\omega\varepsilon_p}{\Gamma^2 c^2}\frac{\partial E_{0z}}{\partial r}, \tag{5}$$

where I_0 is the zeroth-order modified Bessel function, c is the speed of light in vacuum, and Γ is obtained from

$$\Gamma^2 = \beta^2 - \left(\frac{\omega}{c}\right)^2 \varepsilon_p. \tag{6}$$

The numerical results given in the following sections are computed for values of the dielectric permittivity, tube radius and thickness, metallic waveguide radius given in the caption to Fig. 1, and a frequency of 210

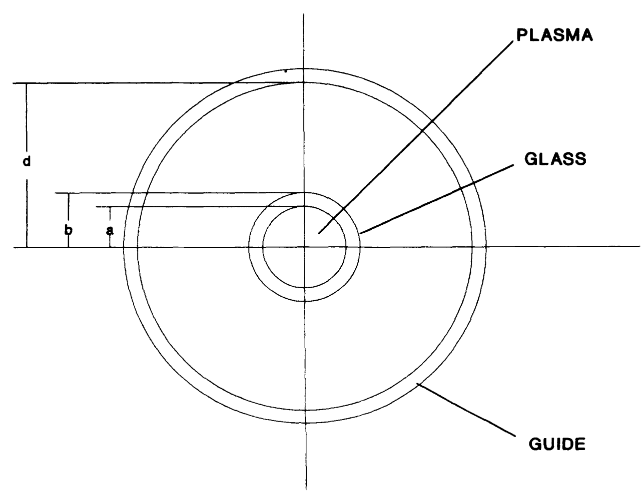

FIG. 1. Scheme employed: $\varepsilon_g = 4$, a=3 mm, b=8 mm, and d=40 mm.

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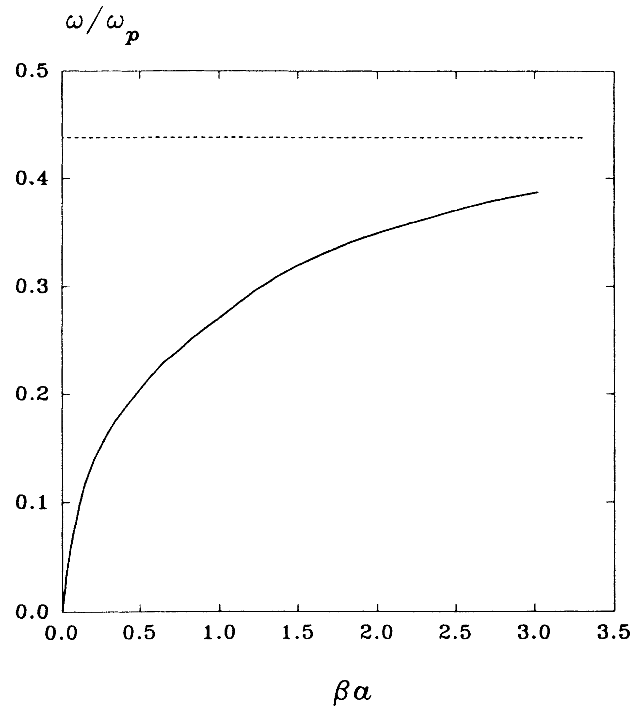

FIG. 2. Dispersion relation of the azimuthally symmetric surface wave propagating over a cylindrical plasma column, computed for the scheme given in Fig. 1 and for 210 MHz. The dashed line is the limiting value obtained when $\beta \to \infty$.

MHz. Figure 2 shows the variation of βa (a is the tube radius) as a function of ω/ω_p for 210 MHz; the dashed line represents the asymptotic value of ω/ω_p as β approaches infinity $[\omega/\omega_p \to 1/(1+\varepsilon_g)^{1/2}]$. We have chosen these numerical values because they are usual values in the experiences with rf surface-wave discharges, nevertheless, this model can be applied to other conditions.

III. SOLUTION OF THE VLASOV EQUATION

If discrete particle correlations are negligible, the electron distribution $f(\vec{r},\vec{v},t)$ evolves according to the Vlasov equation, which may be expressed by

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{r}} - \frac{e}{m}(\vec{E} + \vec{v}\times\vec{B})\cdot\frac{\partial}{\partial \vec{v}}\right) f(\vec{r},\vec{v},t) = 0, \quad (7)$$

where e and m are the charge and the mass of the electron and the fields are given by (2)–(4).

Let $f = f_0 + f_1$, where f_0 is the unperturbed part of the distribution and f_1 is the perturbation caused by the surface wave. Equation (7) can be approximated by its linearized version

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{r}} = \frac{e}{m}(\vec{E} + \vec{v}\times\vec{B})\cdot\frac{\partial f_0}{\partial \vec{v}}. \quad (8)$$

This equation may be solved by the method of characteristics [11] with the specular reflection hypothesis at the surrounding glass tube [9]. We obtain the following expression for f_1:

$$\begin{aligned} f_1(\vec{r},\vec{v},t) &= g_1(\vec{v})\cos\beta(z - v_z t) \\ &+ \frac{e}{T_e} f_0[(\Gamma_{rc} + \Gamma_{zs})\cos\beta(z - v_z t) \\ &+ (\Gamma_{rs} + \Gamma_{zc})]\sin\beta(z - v_z t), \end{aligned} \quad (9)$$

with

$$\Gamma_{is}(\vec{r},\vec{v},t) = \int_0^t dt_0 E_i(r_0) v_{0i} \sin(\omega - \beta v_z)t_0, \qquad i = r, z$$
$$\Gamma_{ic}(\vec{r},\vec{v},t) = \int_0^t dt_0 E_i(r_0) v_{0i} \cos(\omega - \beta v_z)t_0, \quad (10)$$

where $g_1(\vec{v})$ is an arbitrary function of the velocity, and $\vec{r}_0$ and $\vec{v}_0$ are the particle orbits, in absence of fields, which are taken to satisfy

$$\vec{r}_0(t_0 = t) = \vec{r}, \qquad \vec{v}_0(t_0 = t) = \vec{v}. \quad (11)$$

If we call t_{bk} the time at which the breakdown time of the linear solution occurs, we can obtain that [9]

$$t_{bk} = \left(\frac{m}{e\beta E}\right)^{1/2} \overline{\Lambda}, \quad (12)$$

where $\overline{\Lambda}$ is a function of the electron temperature and the ω/ω_p, shown in Fig. 3.

We can extend the theory of collisionless damping to times greater than t_{bk}. To see that, we rewrite (7) in the form

$$\frac{d}{dt_0} f(\vec{r}_0(t_0), \vec{v}(t_0), t_0) = 0, \quad (13)$$

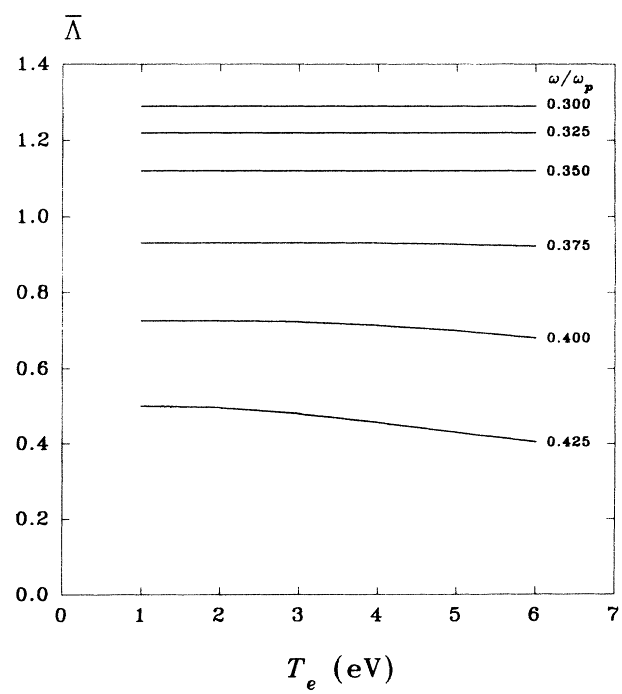

FIG. 3. Radial average of Λ as a function of electron temperature for several values of ω/ω_p.

where $\vec{r}_0(t_0)$ and $\vec{v}_0(t_0)$ are the particle orbits in the electromagnetic field,

$$m\ddot{z}_0 = -e(E_z - v_{0r}B_\phi), \tag{14}$$

$$m\ddot{r}_0 = \frac{l^2}{mr_0^3} - e(E_r - v_{0z}B_\phi) - \frac{\partial\phi_{\rm wall}}{\partial r_0}, \tag{15}$$

$$mr_0v_{0\phi} = l = \text{const}, \tag{16}$$

where $\phi_{\rm wall}$ is the electron-wall interaction potential of the surrounding glass tube and l is the angular momentum of the electron. We suppose that the wall is insulated and becomes negatively charged, owing to the greater mobility of the electrons. The orbits $\vec{r}_0(t_0)$ and $\vec{v}_0(t_0)$ are taken to satisfy

$$\vec{r}_0(t_0 = t) = \vec{r}, \qquad \vec{v}_0(t_0 = t) = \vec{v}. \tag{17}$$

Equation (13) simply states that f is a constant following the particle trajectories, and may be integrated to give

$$f(\vec{r}_0, \vec{v}_0, t_0) = \text{const}. \tag{18}$$

Evaluating this equation at $t_0 = t$ and $t_0 = 0$ and making use of (17) we obtain

$$f(\vec{r}, \vec{v}, t) = f(\vec{r}_0(t_0 = 0), \vec{v}_0(t_0 = 0), t_0 = 0). \tag{19}$$

From (9) it follows that the initial distribution can be written as

$$f(\vec{r}_0(t_0 = 0), \vec{v}_0(t_0 = 0), t_0 = 0)$$

$$= f_0(\vec{v}_0(t_0 = 0)) + g_1(\vec{v}_0(t_0 = 0)) \cos\beta z_0(t_0). \tag{20}$$

Therefore the solution of the Vlasov equation is reduced to the determination of $\vec{r}_0(t_0 = 0)$ and $\vec{v}_0(t_0 = 0)$ from (14)–(17).

IV. ELECTRON DYNAMICS

For the conditions studied in this paper it is readily verified that $B_\phi v_{\rm th}/E_z \ll 1$ and $B_\phi v_{\rm th}/E_r \ll 1$, where $v_{\rm th}$ is the thermal velocity. Therefore (14) and (15) may be expressed by

$$m\ddot{z}_0 = -eEI_0(\Gamma r_0)\sin(\omega t_0 - \beta z_0), \tag{21}$$

$$m\ddot{r}_0 = \frac{l^2}{mr_0^3} - \frac{e\beta E}{\Gamma} I_1(\Gamma r_0)\cos(\omega t_0 - \beta z_0) - \frac{\partial\phi_{\rm wall}}{\partial r_0}, \tag{22}$$

where I_1 is the first-order modified Bessel function.

Transforming to the wave frame, we introduce the new independent variables z_0' and v_{0z}',

$$z_0' = z_0 - \omega t_0/\beta - \pi/\beta, \quad v_{0z}' = v_{0z} - \omega/\beta. \tag{23}$$

Equations (21) and (22) will be expressed as

$$m\ddot{z}_0' = -eEI_0(\Gamma r_0)\sin\beta z_0', \tag{24}$$

$$m\ddot{r}_0 = \frac{l^2}{mr_0^3} - \frac{e\beta E}{\Gamma} I_1(\Gamma r_0)\cos\beta z_0' - \frac{\partial\phi_{\rm wall}}{\partial r_0}. \tag{25}$$

In the linear model [9] we found that the specular reflection implies that the electron exhibits a radial periodic motion of period T, which may be expressed as

$$T = 2(a^2 - \sigma^2)^{1/2}/v_\perp, \quad \sigma = v_\phi r/v_\perp,$$

$$v_\perp = (v_r^2 + v_\phi^2)^{1/2}, \tag{26}$$

where a is the tube radius. The functions I_0 and I_1 may be expressed as a Fourier series. For the present purposes these functions can be approximated in (24) and (25) by the first term ($n = 0$) of the series

$$X_0 = \frac{1}{T}\int_0^T I_0(\Gamma r_0)dt_0, \; X_1 = \frac{1}{T}\int_0^T I_1(\Gamma r_0)dt_0. \tag{27}$$

The integrations are computed along the unperturbed trajectory in the field-free equilibrium. The contribution for $n > 0$ is always less than 15% of the approximation $n = 0$ [9].

Therefore (24) may be expressed in the form

$$m\ddot{z}_0' = -eEX_0 \sin\beta z_0'. \tag{28}$$

This equation coincides formally with the unidimensional motion equation for an electron in the presence of a purely sinusoidal traveling wave [12].

Defining the axial trapping time τ according to

$$\tau = \left(\frac{m}{e\beta EX_0}\right)^{1/2}, \tag{29}$$

Eq. (28) may be integrated once to give

$$\dot{\psi}_0^2 = \frac{1}{\delta^2\tau^2}(1 - \delta^2\sin^2\psi_0), \tag{30}$$

where

$$\psi_0 = \frac{1}{2}\beta z_0', \qquad \delta^2 = \frac{2eEX_0/\beta}{W_z' + eEX_0/\beta}. \tag{31}$$

The case $W_z' > eEX_0/\beta$ ($\delta^2 < 1$) corresponds to the axial untrapped electron orbits and the case $W_z' < eEX_0/\beta$ ($\delta^2 > 1$) corresponds to the axial trapped electron orbits.

The solution (30) may be obtained in terms of the elliptic integral $F(\psi_0;\delta)$ [13]. The orbit quadrature gives

$$(t_0 - t)/\delta\tau = F(\psi_0;\delta) - F(\psi;\delta) \quad (\delta^2 < 1), \tag{32}$$

$$(t_0 - t)/\delta\tau = F(\alpha_0;1/\delta) - F(\alpha;1/\delta) \quad (\delta^2 > 1), \tag{33}$$

where

$$\begin{gathered} \sin\alpha_0 = \delta\sin\psi_0, \qquad \psi = \psi_0(t_0 = t), \\ \alpha = \alpha_0(t_0 = t) = \sin^{-1}(\delta\sin\psi). \end{gathered} \tag{34}$$

According to (32) and (33) we obtain

$$\sin\psi_0 = \text{sn}\; u_< \qquad (\delta^2 < 1), \tag{35}$$

$$\sin\psi_0 = \frac{1}{\delta}\text{sn}\; u_> \qquad (\delta^2 > 1), \tag{36}$$

with

$$u_< = (t_0 - t)/\delta\tau + F(\psi;\delta), \tag{37}$$

$$u_> = (t_0 - t)/\tau + F(\alpha;1/\delta), \tag{38}$$

where sn is a Jacobian elliptic function [13]. The radial evolution of the electron may be expressed in the form

$$m\ddot{r}_0 = \frac{l^2}{mr_0^3} - \frac{eE\beta X_1}{\Gamma}\cos\beta z_0' - \frac{\partial\phi_{\rm wall}}{\partial r_0}. \tag{39}$$

We note that $(l^2/mr_0^3)/(eE\beta X_1/\Gamma) \ll 1/(\omega\tau)^2$. The implications of this result are significant. By neglecting terms down by an order of $1/\omega\tau$, we find for the radial evolution the same equation obtained in the linear analysis [9]. Therefore, in this approximation the electron will exhibit a radial periodic motion of period $T_r \sim a/v_\perp$, where a is the tube radius and $v_\perp$ is the transversal velocity.

The inequality $1/\omega\tau \ll 1$ determines the range of values of E. It follows that $E \ll m\omega^2/e\beta$ ($\sim 10^4$ V/m for the conditions given in Sec. II).

V. DERIVATION OF THE NONLINEAR COLLISIONLESS DAMPING

To obtain the rate of increase of kinetic energy, we need $\partial f/\partial t$ where f is given in (20),

$$\frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial \vec{v}_0'}\cdot\frac{\partial \vec{v}_0'}{\partial t} + \frac{\partial g_1}{\partial \vec{v}_0'}\cdot\frac{\partial \vec{v}_0'}{\partial t}\cos\beta z_0' - g_1\beta\frac{\partial z_0'}{\partial t}\sin\beta z_0', \tag{40}$$

where $z_0' = z_0'(t_0 = 0)$ and $\vec{v}_0' = \vec{v}_0'(t_0 = 0)$. It is readily verified that $\partial z_0'/\partial t_0 = -v_{0z}'$; then (40) may be rewritten

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f_0}{\partial \vec{v}_0'} + \frac{\partial g_1}{\partial \vec{v}_0'}\cos\beta z_0'\right)\cdot\frac{\partial \vec{v}_0'}{\partial t} + g_1\beta v_{0z}'\sin\beta z_0'. \tag{41}$$

By neglecting terms down by an order of $1/\omega\tau$ we can write (41) as

$$\frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial \vec{v}_0'}\cdot\frac{\partial \vec{v}_0'}{\partial t}. \tag{42}$$

To find the rate of absorbed wave energy per electron via collisionless damping, one first calculates the rate of increase of kinetic energy,

$$\frac{\partial W_k}{\partial t} = \frac{m}{2}\int v^2\frac{\partial f}{\partial t}d\vec{v}. \tag{43}$$

The rate of increase of kinetic energy per electron, θ, will be

$$\theta(r,t) = \frac{1}{n_0}\left\langle\frac{\partial W_k}{\partial t}\right\rangle, \tag{44}$$

where n_0 is the electron density and $\partial W_k/\partial t$ is averaged on the wavelength λ.

Substituting (42) and (43) into (44) and making use of the results obtained in Sec. IV, we may then obtain the following expression for θ:

$$\theta(r,t) = E^2\int_{-\lambda/2}^{\lambda/2}\frac{dz'}{\lambda}\int d\vec{v}_\perp dv_z'\Phi_1 v_z'\sin\beta z_0'(t_0 = 0), \tag{45}$$

where the function Φ_1 is defined as

$$\Phi_1 = -\frac{16e^2}{\pi\beta T_e}v_{\rm ph}^2\left(\frac{2\pi T_e}{m}\right)^{-3/2}\exp\left(-\frac{mv_{\rm ph}^2}{2T_e}\right) \times X_0^2\exp\left(-\frac{mv_\perp^2}{2T_e}\right), \tag{46}$$

with $v_{\rm ph} = \omega/\beta$.

To solve (45) we parallel the analysis of O'Neil [12] and make use of the independent variables (ψ,δ). Equation (45) reduces to

$$\theta(r,t) = E^2\int d\vec{v}_\perp\Phi_1\Phi_2, \tag{47}$$

with

$$\Phi_2 = \int_0^1\frac{d\delta}{\delta^3}\int_0^{\pi/2}d\psi\sin 2\psi_0(t_0 = 0) + \int_1^\infty\frac{d\delta}{\delta^3}\int_0^{\sin^{-1}(1/\delta)}d\psi\sin 2\psi_0(t_0 = 0). \tag{48}$$

By using the double angle formula of the Jacobi elliptic functions and expressing the integrands as series expansions in terms of the parameter $q = e^{\pi K'/K}$ and the arguments $\nu_< = \pi/2K\delta\tau$ and $\nu_> = \pi/2K\tau$ [13] we obtain

$$\Phi_2 = -\sum_{n=0}^{\infty}2\pi\int_0^1 d\delta\left(-\frac{2n\pi^2\sin 2n\nu_< t}{\delta^5K^2(1+q^{2n})(1+q^{-2n})} + \frac{(2n+1)\pi^2\delta\sin(2n+1)\nu_> t}{K^2(1+q^{2n+1})(1+q^{-2n-1})}\right), \tag{49}$$

where $K = F(\pi/2;\delta)$ and $K' = F(\pi/2;(1-\delta^2)^{1/2})$.

It is useful to compare expression (47) with the result obtained in the lineal model [9], θ_L, for $t \ll \tau$. Figure 4 shows the radial variation of θ/θ_L.

We consider the expression for the nonlinear damping coefficient γ ($\gamma = N\overline{\theta}/U$ where U is the total energy, N is the number of electrons, and $\overline{\theta}$ is the radial average of θ). For $t \ll \tau$ one can show that this reduces to the linear coefficient; this one is shown in Fig. 5. For $t \gg \tau$ the integration in (47) causes the fast oscillation to phase mix to zero, then

$$\gamma \to 0 \qquad (t \to \infty). \tag{50}$$

In addition it may be shown that the integral damping coefficient Θ may be expressed as

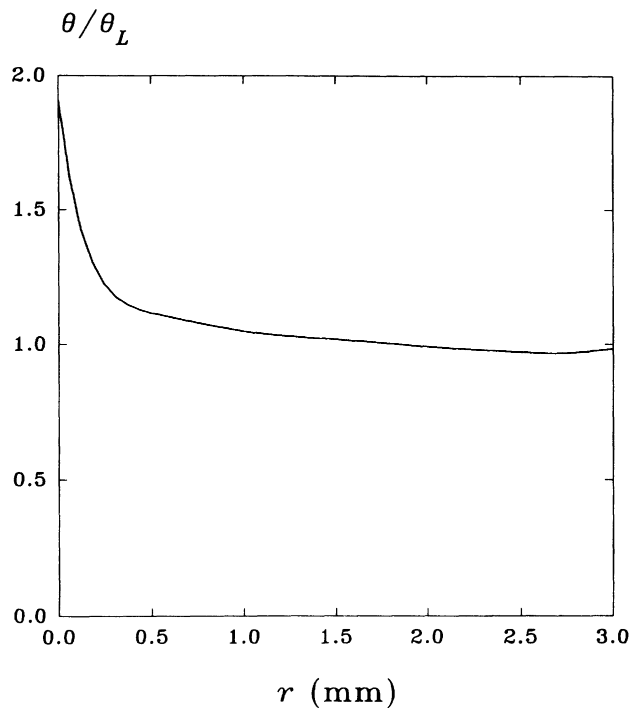

FIG. 4. Radial dependence of θ/θ_L for several values of ω/ω_p.

$$\Theta = \int_0^\infty \gamma(t)\, dt = \frac{64}{\pi}\zeta\eta\left(\frac{m}{e\beta E}\right)^{1/2}\gamma_L, \qquad (51)$$

where γ_L is the linear damping coefficient, the function η is shown in Fig. 6 for several values of ω/ω_p, and ζ is shown to have the value

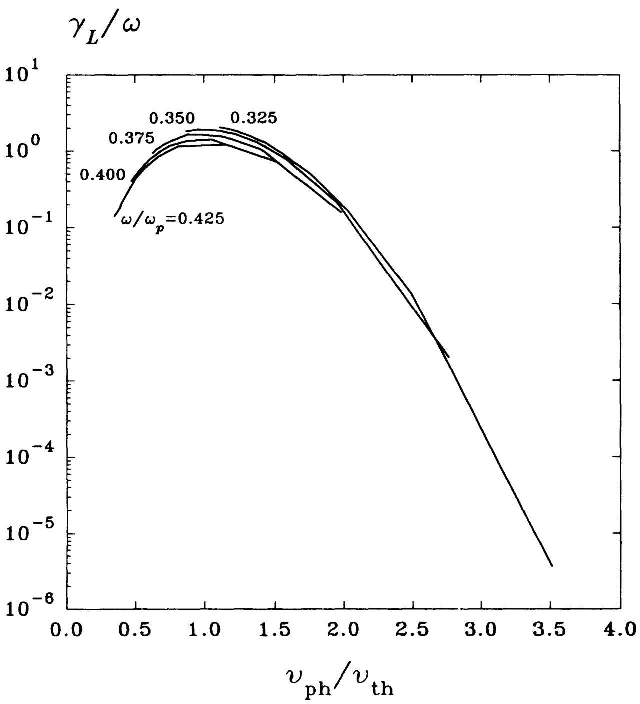

FIG. 5. Ratio γ_L/ω as a function of the ratio of phase velocity to thermal velocity.

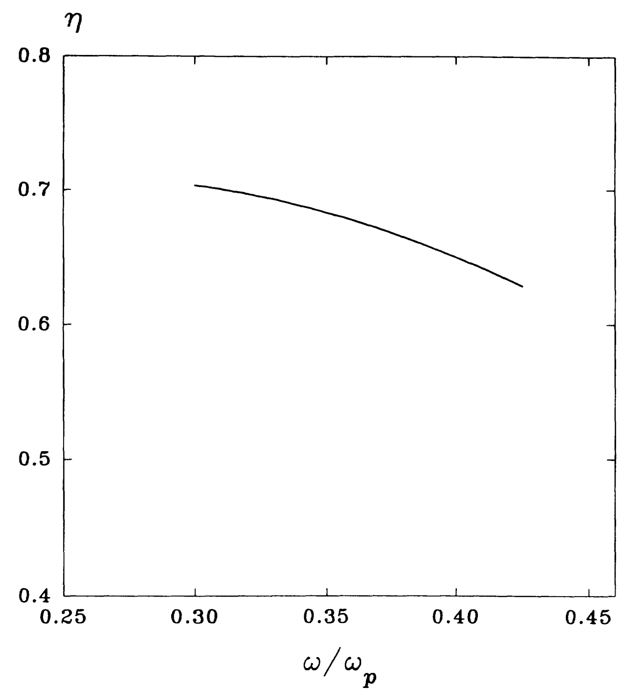

FIG. 6. Dependence of η as a function of ω/ω_p.

$$\zeta = \int_0^1 d\delta \left[\frac{1}{\delta^4}\left(\frac{E}{\pi} - \frac{\pi}{4K}\right) + \frac{\delta}{\pi}\left[E + (\delta^2 - 1)K\right]\right], \qquad (52)$$

where it is expressed as a function of the complete elliptic integrals of the first and second kinds [12]. The function η is independent of the electron temperature under the conditions studied in this paper.

It follows from (51) that $\Theta \sim O(t_{\rm bk}\gamma_L)$, consequently, the amplitude of the electric field E will be constant under the condition $t_{\rm bk}\gamma_L \ll 1$. This condition is satisfied as we saw in the analysis of the electron radial motion in Sec. IV.

VI. CONCLUSIONS

In this paper the nonlinear surface-wave–particle interactions in a cylindrical plasma are analysed, using the specular reflection hypothesis in the resolution of the Vlasov equation. An expression for the nonlinear collisionless damping coefficient has been obtained as a function of the critical parameters. For $t \ll t_{\rm bk}$ (breakdown time of the linear solution) this coefficient reduces to a linear result and it is zero as t approaches infinity. The integral damping coefficient Θ has also been calculated and it was shown that $\Theta \sim O(t_{\rm bk}\gamma_L)$. It follows from this result that the change in the amplitude of the electric field is small since $t_{\rm bk}\gamma_L \ll 1$. Then we may conclude from this analysis that, under appropriate conditions, the collisionless damping must be taken into account in the surface-wave propagation along a cylindrical plasma column.

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